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A LAPLACE–SERIES AND FIXED-POINT FRAMEWORK FOR A FAMILY OF NONLINEAR FREDHOLM INTEGRAL EQUATIONS WITH POWER-LAW NONLINEARITY

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ABSTRACT

We focus on a family of nonlinear Fredholm integral equations (NFIE) of the second kind [2] with separable kernels and power-law nonlinearity. By combining Laplace transform techniques, power-series representations, and fixed-point theory, the integral equation is rigorously reduced to a finite-dimensional nonlinear algebraic equation. The solutions exist using Schauder's fixed-point theorem, while local uniqueness is obtained via the Banach contraction principle. A complete parameter-dependent analysis is presented, identifying conditions under which solutions exist, are unique, or exhibit multiplicity. The results generalize known quadratic cases to arbitrary positive powers and provide a unified and transparent analytical framework.

Keywords: *Nonlinear Fredholm integral equation; Laplace–series method; power-law nonlinearity; existence and uniqueness; bifurcation analysis.*

1 INTRODUCTION

NFIE of the second kind arise in a wide range of applications in physics, applied mathematics and engineering [2, 3]. While numerical methods are commonly employed, analytical investigations remain crucial for understanding solution structure, multiplicity, and parameter dependence.

For equations with separable kernels, classical techniques allow dimensional reduction; however, in many works the solution structure is assumed a priori. The objective of this paper is to present a unified and fully justified analytical framework that combines Laplace transforms, series methods, and fixed-point theory for a family of NFIE with power-law nonlinearity [2].

The novelty of the current work lies not in introducing a new equation, but in providing a rigorous synthesis of analytical tools together with a complete parameter-dependent classification of solutions.

2 PROBLEM STATEMENT AND ASSUMPTIONS [8]

Let consider the NFIE

$$u(x) = x + \lambda \int_0^1 x\, t\, u^p(t)\, dt, \qquad x \in [0,1], \tag{1}$$

where

$$p > 0, \qquad \lambda > 0.$$

We seek solutions in the Banach space $C[0,1]$ which is equipped with the supremum norm

$$\| u \|_\infty = \max_{x\in[0,1]} |u(x)|.$$

Throughout the paper, we assume

$$u(x) \geq 0 \quad \text{for all } x \in [0,1],$$

which is natural in view of the solution structure obtained below and is required when p is non-integer.

3 STRUCTURAL REDUCTION

Since the kernel $K(x,t) = xt$ is separable, the equation will be,

$$u(x) = x + \lambda x \int_0^1 t\, u^p(t)\, dt. \tag{2}$$

Define

$$k = \int_0^1 t\, u^p(t)\, dt. \tag{3}$$

Then

$$u(x) = x(1 + \lambda k). \tag{4}$$

This representation suggests that any solution must be linear in x, a fact rigorously justified in the next section.

4 LAPLACE–SERIES DERIVATION

Let $U(s) = \mathcal{L}\{u(x)\}$. Taking the Laplace transform gives

$$U(s) = (1 + \lambda k)\mathcal{L}\{x\} = \frac{1+\lambda k}{s^2}. \tag{5}$$

Applying the inverse Laplace transform yields

$$u(x) = (1 + \lambda k)x. \tag{6}$$

To confirm consistency with series methods, assume

$$u(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{7}$$

Termwise Laplace transformation and coefficient comparison show that all coefficients vanish except c_1, yielding

$$u(x) = c_1 x, \qquad c_1 = 1 + \lambda k. \tag{8}$$

Thus, the Laplace and series approaches are fully consistent.

5 REDUCTION TO A NONLINEAR ALGEBRAIC EQUATION [5, 6]

Substituting $u(x) = ax$ into the definition of k, we obtain

$$k = \int_0^1 t(at)^p \, dt = a^p \int_0^1 t^{p+1} \, dt = \frac{a^p}{p+2}. \tag{9}$$

Hence,

$$a = 1 + \frac{\lambda}{p+2} a^p, \tag{10}$$

or equivalently,

$$\lambda a^p - (p+2)a + (p+2) = 0. \tag{11}$$

6 EXISTENCE OF SOLUTIONS

Define the operator $T: C[0,1] \to C[0,1]$ by

$$(Tu)(x) = x + \lambda x \int_0^1 t\, u^p(t) \, dt. \tag{12}$$

For $R > 0$, define

$$B_R = \{u \in C[0,1]: \| u \|_\infty \leq R\}.$$

LEMMA 6.1 (INVARIANT BALL)

There exists $R > 0$ *such that* $T(B_R) \subset B_R$.

Proof. For $u \in B_R$,

$$\| Tu \|_\infty \leq 1 + \frac{\lambda R^p}{2}.$$

Choosing R such that $1 + \frac{\lambda R^p}{2} \leq R$ ensures invariance.

THEOREM 6.2 (EXISTENCE)

The NFIE admits at least one solution in $C[0,1]$.

Proof. The operator T is continuous and compact [1, 4] by the Arzelà–Ascoli theorem and maps the closed, convex set B_R into itself. Schauder's fixed-point theorem gives guarantees the existence of a fixed point.

7 LOCAL UNIQUENESS OF SOLUTIONS

THEOREM 7.1 (LOCAL UNIQUENESS)

If $\lambda\, p\, R^{p-1} < 2$, then the NFIE [2, 8] has at most one solution in B_R.

Proof. For $u, v \in B_R$,

$$|u^p(t) - v^p(t)| \leq pR^{p-1}|u(t) - v(t)|.$$

Hence,

$$\| Tu - Tv \|_\infty \leq \frac{\lambda p R^{p-1}}{2} \| u - v \|_\infty.$$

If $\lambda p R^{p-1} < 2$, the operator T is a contraction. Uniqueness follows from Banach's fixed-point theorem.

8 PARAMETER-DEPENDENT EXISTENCE OF SOLUTIONS [7]

Here, we find the values of the parameters p and λ for which the NFIE admits solutions. As shown in the previous sections, solutions of the integral equation are characterized by the nonlinear algebraic equation [9, 10]

$$\lambda a^p - (p+2)a + (p+2) = 0, \qquad a > 0, \tag{13}$$

where $p > 0$ and $\lambda > 0$.

Let

$$F(a) = \lambda a^p - (p+2)a + (p+2).$$

Clearly, F is continuous on $(0, \infty)$.

8.1 CASE $0 < p < 1$

For $0 < p < 1$, we observe that

$$\lim_{a \to 0^+} F(a) = p + 2 > 0, \qquad \lim_{a \to \infty} F(a) = -\infty.$$

Hence, by the Intermediate Value Theorem, equation (13) admits at least one non-negative solution for every $\lambda > 0$.

$$0 < p < 1,\ \lambda > 0 \quad \Longrightarrow \quad \text{at least one solution exists.}$$

8.2 CASE $p = 1$

When $p = 1$, equation (13) reduces to

$$(\lambda - 3)a + 3 = 0.$$

Thus,

$$a = \frac{3}{3-\lambda}.$$

If $\lambda \neq 3$, there exists a unique positive solution, whereas no solution exists when $\lambda = 3$.

$$p = 1,\ \lambda \neq 3 \Rightarrow \text{unique solution}, \qquad p = 1,\ \lambda = 3 \Rightarrow \text{no solution}.$$

8.3 CASE $p > 1$

For $p > 1$, we have

$$F(0) = p + 2 > 0, \qquad \lim_{a\to\infty} F(a) = +\infty.$$

In this case, solutions exist only if the global minimum of F is non-positive.
The unique critical point of F is given by

$$a_* = \left(\frac{p+2}{\lambda p}\right)^{\frac{1}{p-1}},$$

and the corresponding critical parameter value is

$$\lambda_c = \frac{(p+2)(p-1)^{p-1}}{p^p}. \tag{14}$$

Accordingly, equation (13) admits:

- two non-negative solutions if $0 < \lambda < \lambda_c$,
- exactly one (double) solution if $\lambda = \lambda_c$,
- no solution if $\lambda > \lambda_c$.

8.4 SUMMARY OF RESULTS

The above analysis can be summarized as follows:

- For $0 < p < 1$, solutions exist for all $\lambda > 0$.
- For $p = 1$, a unique solution exists for $\lambda \neq 3$, while no solution exists for $\lambda = 3$.
- For $p > 1$, solutions exist if and only if $\lambda \leq \lambda_c$, where λ_c is given by (14).

This classification explains the dependence of existence and multiplicity of solutions on the parameters p and λ.

9 MATLAB CODE SECTION [11]

```
clear; clc; close all;

lambda = input('Enter value of lambda (e.g. 0.7): ');

p = input('Enter power p (e.g. 2, 3, 1.5): ');

f = @(a) lambda*a.^p - (p+2)*a + (p+2);

initial_guesses = linspace(0.01, 20, 50); roots_found = [];

for i = 1:length(initial_guesses) try a0 = fzero(f, initial_guesses(i));

if isreal(a0) a0 > 0 roots_found(end+1) = a0; end catch end end

a_vals = unique(round(roots_found,6));

fprintf(' positive solutions for a:'); disp(a_vals.')

if isempty(a_vals) error('No real positive solutions found for given parameters.');

end

x = linspace(0,1,200);

figure; hold on; grid on;

for k = 1:length(a_vals) u = a_vals(k)*x; plot(x, u, 'LineWidth', 2, ... 'DisplayName', sprintf('u(x) = end

xlabel('x'); ylabel('u(x)'); title(sprintf('Solutions for p = legend('show','Location','northwest');
```

```
fprintf(' verification:');

t = linspace(0,1,2000);

for k = 1:length(a_vals) a = a_vals(k); u_t = a*t; integrand = t .* (u_t.^p);

K = trapz(t, integrand); rhs = x .* (1 + lambda*K); lhs = a*x; err = max(abs(lhs - rhs));

fprintf('a = a, K, err);

end
```

10 CONCLUSION

A family of NFIE with power-law nonlinearity has been analyzed using Laplace transforms, series methods, and fixed-point theory. The problem is reduced to a nonlinear algebraic equation, existence and local uniqueness are rigorously established, and bifurcation behavior is completely characterized. The framework generalizes known quadratic cases and provides a foundation for further extensions.

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