

A REVISIT TO HIGHER VARIATIONS OF A FUNCTIONAL

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Abstract: -

Two definitions of higher variations of a functional can be found in the literature of variational principles or calculus of variations, which differ by only a positive coefficient number. At first glance, such a discrepancy between the two definitions seems to be purely due to a definition-style preference, as when they degenerate to the first variation it leads to the same result. The use of higher (especially second) variations of a functional is for checking the sufficient condition for the functional to be a minimum (or maximum), and both definitions also lead to the same conclusion regarding this aspect. However, a close theoretical study in this paper shows that only one of the two definitions is appropriate and the other is advised to be discarded. A theoretical method is developed to derive the expressions for higher variations of a functional, which is used for the above claim.

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1 INTRODUCTION

Calculus of variations or variational principles has been widely used in engineering and physics by formulating the problems in terms of extremum principles for certain functionals. For elasticity problems in continuum mechanics, this comes to the familiar example of principles of minimum potential energy and principles of minimum complementary energy, which can be generalized with the use of Lagrange multipliers leading to Hellinger-Reissner (HR) and Hu-Washizu (HW) variational principles, see Washizu [23]. The direct method of variational principles such as Rayleigh-Ritz method and Galerkin’s method is powerful and has formed the basis for development of finite element (FE) formulations, see Bathe[1] and Zienkiewicz[25]. In some cases only the necessary conditions for a certain functional to be a minimum or maximum are needed, i.e. using the stationary property of the functional, and the sufficient conditions could be justified via common sense. However, the study of necessary conditions is needed when common sense fails to justify them easily. This leads to the consideration of higher variations of a functional, especially the second variations, see [3,4,18–20] for the application of second variations in mechanics. The first who considered the second variation and sufficient conditions in calculus of variations is due to Legendre in 1786 according to Todhunter[22], which was then further developed by Jacobi, Clebsch, Mayer, Weierstrass, Kneser, Hilbert etc, see [7,9,17] for a historical perspective. There are two forms of the second variations (or accordingly higher variations) for a functional, which can be broadly found in the literature and they differ by a positive coefficient $n!$, where n is the variation order. Take a functional I as an example, and consider its second variation, with

$$I = \int_a^b F(u, u', x) dx \tag{1.1}$$

where

$u = u(x), u' = u'(x) = \frac{du(x)}{dx} = \frac{du}{dx}$, x is the dummy independent variable, and $F(u, u', x)$ is a known function of u, u' and x , and $F(u, u', x)$ is assumed to take whatever smooth requirements in terms of operations throughout this paper. One form of the second variation

$$\delta^2 I = \int_a^b \left(\frac{\partial^2 F}{\partial u^2} \delta u^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u \delta u' + \frac{\partial^2 F}{\partial u'^2} \delta u'^2 \right) dx \tag{1.2}$$

The other form

$$\delta^2 I = \frac{1}{2!} \int_a^b \left(\frac{\partial^2 F}{\partial u^2} \delta u^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u \delta u' + \frac{\partial^2 F}{\partial u'^2} \delta u'^2 \right) dx \tag{1.3}$$

The first form described by Eq. 1.2 was used by Weierstrass in his unpublished lecture notes around 1879 and was favored by Bolza[2], Lanczos [12], Langhaar [13], Kot [11] and Tauchert [21] among many others. The second form in Eq. 1.3 appears in most Chinese literature of variational principles such as Chen [5], Lao [14], Long et al [15] and Hu [10] as well as in notable English texts such as Courant and Hilbert [6], Gelfand and Fomin [8] and MestertonGibbons [16] among others. It is well known that the higher variations is for checking the sufficient condition for a functional to be a minimum or maximum and in this sense the conclusion made based on Eq. 1.2 or Eq. 1.3 does not make a difference since the two values always have the same sign being positive or negative or zero. This is why the difference between the two forms has not called much attention. Among the above mentioned literature, only Lao [14] presented both forms and treated the difference as a purely definition-style preference without further looking into it. The background behind these two definitions will be explored in this paper and it is found only the first form in Eq. 1.2 is consistent, and the second form in Eq. 1.3 has caused confusion and is advised to be discarded.

2 Variational Process in conformity with Eq. 1.2

In this session, the variational process associated with Eq. 1.2 regarding the functional defined by Eq. 1.1 will be reviewed, the first (or first order) variation will be defined in a natural way with Taylor’s series and further expansion of higher variations including second order and even higher order will be explored.

2.1 Variation of u

There is no harm to assume the problem of finding the function $u(x)$ that minimizes the functional I in Eq. 1.1, with prescribed end conditions

$$u(a) = u_a, u(b) = u_b \tag{2.1}$$

For an admissible dependent function $u(x)$, i.e. meeting the end conditions described by Eq. 2.1, Eq. 1.1 yields a numerical value of I , which means that I is a function of function and therefore termed functional. In the calculus of variations, interest is placed on the minimizing curve denoted by $u(x)$. To this end, $u(x)$ is replaced by an admissible neighbour function, or called comparison function $\bar{u}(x)$, and the behavior of the value of $I(u)$ is observed as $u(x)$ changes to $\bar{u}(x)$. The difference between $\bar{u}(x)$ and $u(x)$ is defined to be the variation of $u(x)$ and is denoted by δu , see Fig. 1, thus

$$\delta u \equiv \bar{u}(x) - u(x) \tag{2.2}$$

Sometimes for the convenience of using the differential calculus, the variation of $u(x)$ is represented by

$$\delta u \equiv \epsilon \eta(x) \tag{2.3}$$

where ϵ is a small parameter independent of x and $\eta(x)$ is an arbitrary function which must meet the admissible requirement in Eq. 2.1 so that

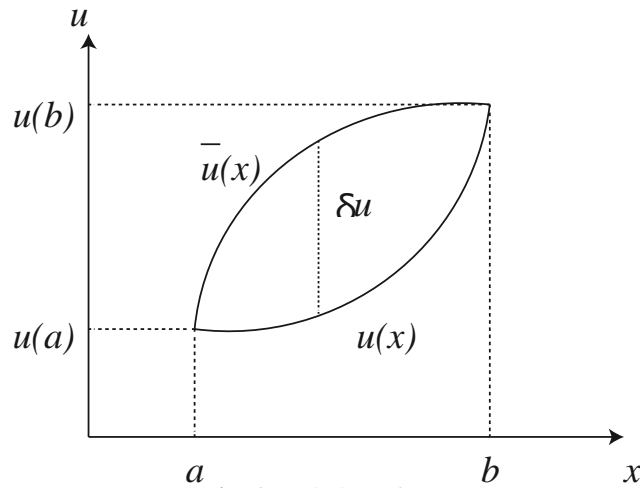


Fig. 1 Variation of $u(x)$

$$\eta(a) = \eta(b) = 0 \tag{2.4}$$

The use of variational operator “ δ ” is becoming more and more popular with researchers and engineers, although Weinstock [24] disapproved the use of it.

2.2 Variation of F

We next consider the behavior of the function $F(u, u', x)$ in the neighbourhood of the minimizing curve $u(x)$, which leads to the natural definition of δF . For a fixed $x, F(u, u', x)$ depends upon u and u' . So $F(u, (u)')$ differs from $F(u, u', x)$ by the increment (sometimes called total variation)

$$\begin{aligned} \Delta F &= F(\bar{u}, (\bar{u})', x) - F(u, u', x) \\ &= F(u + \delta u, (u + \delta u)', x) - F(u, u', x) \\ &= F(u + \delta u, u' + (\delta u)', x) - F(u, u', x) \end{aligned} \tag{2.5}$$

We may expand $F(u + \delta u, u' + (\delta u)', x)$ in a Taylor’s series as

$$\begin{aligned} &F(u + \delta u, u' + (\delta u)', x) \\ &= F(u, u', x) + \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} (\delta u)' + \frac{1}{2!} (\delta u \frac{\partial}{\partial u} + (\delta u)' \frac{\partial}{\partial u'})^2 F + \dots \\ &+ \frac{1}{n!} (\delta u \frac{\partial}{\partial u} + (\delta u)' \frac{\partial}{\partial u'})^n F + \dots \\ &= F(u, u', x) + \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} (\delta u)' + \frac{1}{2!} (\frac{\partial^2 F}{\partial u^2} (\delta u)^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u (\delta u)' + \\ &\frac{\partial^2 F}{\partial u'^2} (\delta u)'^2) + \dots + \frac{1}{n!} (\delta u \frac{\partial}{\partial u} + (\delta u)' \frac{\partial}{\partial u'})^n F + \dots \end{aligned} \tag{2.6}$$

The first variation of F is defined as

$$\delta F \equiv \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} (\delta u)' \tag{2.7}$$

The second variation of F is defined as

$$\delta^2 F \equiv \frac{\partial^2 F}{\partial u^2} (\delta u)^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u (\delta u)' + \frac{\partial^2 F}{\partial u'^2} (\delta u)'^2 \tag{2.8}$$

Similarly, the $n - th$ variation of F is defined as

$$\delta^n F \equiv (\delta u \frac{\partial}{\partial u} + (\delta u)' \frac{\partial}{\partial u'})^n F \tag{2.9}$$

As most variational principles refer to the first variation, in this paper higher variations refer to second variations and above. For an exceptional case

$$F(u, u', x) = u' \tag{2.10}$$

Inserting Eq. 2.10 into eq Eq. 2.7, leads to a corollary:

$$\delta u' = (\delta u)' \tag{2.11}$$

, which means the variational operator and differential operator are interchangeable. This corollary has been stated in many textbooks on variational principles without a strict proof.

Note that we write

$$(\delta u)^2 = \delta u^2, (\delta u)'^2 = \delta u'^2 \tag{2.12}$$

By using Eq. 2.11 and, Eq. 2.12, the Eq. 2.7 to Eq. 2.9 can now be respectively written as

$$\delta F = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \tag{2.13}$$

$$\delta^2 F = \frac{\partial^2 F}{\partial u^2} \delta u^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u \delta u' + \frac{\partial^2 F}{\partial u'^2} \delta u'^2 \tag{2.14}$$

$$\delta^n F = \left(\delta u \frac{\partial}{\partial u} + \delta u' \frac{\partial}{\partial u'} \right)^n F \tag{2.15}$$

By Eq. 2.14, if we set

$$F(u, u', x) = u \tag{2.16}$$

Then obviously

$$\delta(\delta u) = \delta^2 u = 0 \tag{2.17}$$

Similarly, if we set $F(u, u', x) = u'$ (Eq. 2.10) then by Eq. 2.14, we have

$$\delta(\delta u') = \delta^2 u' = 0 \tag{2.18}$$

Therefore, the Eq. 2.17 and Eq. 2.18 are the corollaries.

From Eq. 2.13, it is easy to shown (proof omitted here):

$$\delta(F_1 + F_2) = \delta F_1 + \delta F_2 \tag{2.19}$$

$$\delta(F_1 F_2) = F_2 \delta F_1 + F_1 \delta F_2 \tag{2.20}$$

With Eq. 2.17-Eq. 2.20, now it is possible to get $\delta(\delta F)$ and compare this with $\delta^2 F$ in Eq. 2.14. The result is confirmative, as shown below

$$\begin{aligned} \delta(\delta F) &= \delta \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \right) \\ &= \delta \left(\frac{\partial F}{\partial u} \right) \delta u + \delta \left(\frac{\partial F}{\partial u'} \right) \delta u' \\ &= \left(\frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u} \right) \delta u + \frac{\partial}{\partial u'} \left(\frac{\partial F}{\partial u} \right) \delta u' \right) \delta u + \left(\frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u'} \right) \delta u + \frac{\partial}{\partial u'} \left(\frac{\partial F}{\partial u'} \right) \delta u' \right) \delta u' \\ &= \frac{\partial^2 F}{\partial u^2} \delta u^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u \delta u' + \frac{\partial^2 F}{\partial u'^2} \delta u'^2 \end{aligned} \tag{2.21}$$

Therefore, we have the corollary

$$\delta(\delta F) = \delta^2 F \tag{2.22}$$

In general,

$$\delta(\delta^{n-1} F) = \delta^n F \tag{2.23}$$

This seems obvious, but it will be shown in Section 3 that the types of equation similar to Eq. 2.22 and Eq. 2.23 will fail to exist if the definition of the second variation is with conformity of Eq. 1.3, thus causing inconsistency. We will call the approach for obtaining $\delta^2 F$ from $\delta(\delta F)$ the “ $\delta\delta$ ” approach in this paper. Substitution of the definitions from Eq. 2.7 - Eq. 2.9 into Eq. 2.6 yields

$$\Delta F = \delta F + \frac{1}{2} \delta^2 F + \dots + \frac{1}{n!} \delta^n F + \dots \tag{2.24}$$

2.3 Variation of I

The difference between the minimum value of I and the value of I evaluated for varied curve $u(x)$ may now be written as

$$\begin{aligned} \Delta I &= I(\bar{u}) - I(u) \\ &= \int_a^b F(\bar{u}, (\bar{u})', x) dx - \int_a^b F(u, u', x) dx \\ &= \int_a^b \Delta F dx \end{aligned} \tag{2.25}$$

By virtue of Eq. 2.24, we have

$$\Delta I = \int_a^b \left(\delta F + \frac{1}{2} \delta^2 F + \dots + \frac{1}{n!} \delta^n F + \dots \right) dx \tag{2.26}$$

The first, second and $n - th$ variations of the functional I are respectively defined by Eq. 2.27 - Eq. 2.29.

$$\delta I = \int_a^b \delta F dx \tag{2.27}$$

$$\delta^2 I = \int_a^b \delta^2 F dx \tag{2.28}$$

$$\delta^n I = \int_a^b \delta^n F dx \tag{2.29}$$

Similar to Eq. 2.23, we have in general

$$\delta(\delta^{n-1}I) = \delta^n I \tag{2.30}$$

Substitution of Eq. 2.27 - Eq. 2.29 into Eq. 2.26 yields

$$\Delta I = \delta I + \frac{1}{2}\delta^2 I + \dots + \frac{1}{n!}\delta^n I + \dots \tag{2.31}$$

By requiring $\delta I = 0$, the Euler-Lagrange equation can be derived as shown in Eq. 2.32, which shows a necessary condition in terms of differential equations for the minimizing curve $u(x)$ in order for I to be a minimum. The proof of this can be found in many standard texts on this such as Lanczos [12].

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) = 0 \tag{2.32}$$

3 Variational Process in conformity with Eq. 1.3

Rather than go through all the similar steps in Section 2, we focus on the main steps and changes during the variational process implied by Eq. 1.3.

3.1 Variation of u

The first variation of u is still defined by Eq. 2.2 and Eq. 2.3, which are repeated below.

$$\delta u = \bar{u}(x) - u(x) \text{ (Eq. 2.2)}$$

$$\delta u = \epsilon \eta(x) \text{ (Eq. 2.3)}$$

It will be shown in Section 3.2 that the second variation of u in conformity with Eq. 1.3 is also zero.

3.2 Variation of F

Due to the definition in Eq. 1.3, the set of equations similar to those in Eq. 2.13 to Eq. 2.15 regarding the variations of F will take different forms accordingly. The definition of Eq. 2.13 for the first variation of F is the same, but the Eq. 2.14 and Eq. 2.15 for higher order change to Eq. 3.1 and Eq. 3.2 in order to be in conformity with Eq. 1.3.

$$\tilde{\delta}^2 F \equiv \frac{1}{2!} \left(\frac{\partial^2 F}{\partial u^2} \delta u^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u \delta u' + \frac{\partial^2 F}{\partial u'^2} \delta u'^2 \right) \tag{3.1}$$

$$\tilde{\delta}^n F \equiv \frac{1}{n!} \left(\delta u \frac{\partial}{\partial u} + \delta u' \frac{\partial}{\partial u'} \right)^n F \tag{3.2}$$

It can be seen that the definition of Eq. 3.2 for n -th variation will degenerate to the first variation when $n = 1$. Note that we have used δ^2 and δ^n to represent the second variation and n -th variation in conformity with Eq. 1.3, to distinguish them from those defined in Section 2. For the first variation, there is no difference between δ and $\tilde{\delta}$, and so we have used the symbol δ for the above first variations.

Similar to the procedures in Section 2, by setting $F(u, u', x) = u$ (Eq. 2.16) $F(u, u', x) = u'$ (Eq. 2.10), respectively, and then by inserting these into Eq. 3.1, we still have similar corollaries like Eq. 2.17 and Eq. 2.18,

$$\delta(\delta u) = \delta^2 u = 0 \tag{3.3}$$

$$\delta(\delta u') = \delta^2 u' = 0 \tag{3.4}$$

Now, let's calculate $\delta(\delta F)$.

$$\begin{aligned} \delta(\delta F) &= \delta \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \right) \\ &= \delta \left(\frac{\partial F}{\partial u} \right) \delta u + \delta \left(\frac{\partial F}{\partial u'} \right) \delta u' \\ &= \left(\frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u} \right) \delta u + \frac{\partial}{\partial u'} \left(\frac{\partial F}{\partial u} \right) \delta u' \right) \delta u + \left(\frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u'} \right) \delta u + \frac{\partial}{\partial u'} \left(\frac{\partial F}{\partial u'} \right) \delta u' \right) \delta u' \\ &= \frac{\partial^2 F}{\partial u^2} \delta u^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u \delta u' + \frac{\partial^2 F}{\partial u'^2} \delta u'^2 \\ &= 2! \tilde{\delta}^2 F \end{aligned} \tag{3.5}$$

which means that $\delta(\delta F)$ or $\tilde{\delta}(\delta F)$ is no longer equal to $\delta^2 F$ defined by Eq. 3.1, but equal to $2! \tilde{\delta}^2 F$, which is awkward. Similarly, the form similar to Eq. 2.23 fails to exist for δ^n . Actually, from Eq. 2.15 and Eq. 3.2, we can easily obtain

$$\tilde{\delta}^n F = \frac{1}{n!} \delta^n F \quad \text{or} \quad \delta^n F = n! \tilde{\delta}^n F \quad \text{when} \quad n \geq 1 \tag{3.6}$$

Also, obviously

$$\tilde{\delta}^{n-1}F = \frac{1}{(n-1)!} \delta^{n-1}F \quad \text{when } n \geq 2 \quad (3.7)$$

Therefore,

$$\begin{aligned} \delta(\tilde{\delta}^{n-1}F) &= \tilde{\delta}(\tilde{\delta}^{n-1}F) = \frac{\delta(\delta^{n-1}F)}{(n-1)!} \\ &= \frac{\delta^n F}{(n-1)!} = \frac{n!}{(n-1)!} \tilde{\delta}^n F \quad \text{when } n \geq 2 \end{aligned} \quad (3.8)$$

From Eq. 3.6, it is obvious that

$$\underbrace{\delta\delta \dots (\delta F)}_{n \text{ times}} = \underbrace{\tilde{\delta}\tilde{\delta} \dots (\tilde{\delta} F)}_{n \text{ times}} = \delta^n F = n! \tilde{\delta}^n F \quad (3.9)$$

Although this derivation is done for F , similar process can be made for the functional I . This constitutes the main objection of using Eq. 1.3 as the definition of the second variation. Also, implied by Eq. 1.3, the Eq. 2.24 for the total increment of F , needs to be updated by

$$\Delta F = \delta F + \tilde{\delta}^2 F + \dots + \tilde{\delta}^n F + \dots \quad (3.10)$$

3.3 Variation of I

From Eq. 2.25 and Eq. 3.10, we have

$$\begin{aligned} \Delta I &= I(\bar{u}) - I(u) \\ &= \int_a^b \Delta F dx = \int_a^b (\delta F + \tilde{\delta}^2 F + \dots + \tilde{\delta}^n F + \dots) dx \end{aligned} \quad (3.11)$$

The first, second, $n - th$ variations of I are defined similar to Eq. 2.27 to Eq. 2.29 respectively. So,

$$\delta I = \tilde{\delta} I = \int_a^b \delta F dx \quad (3.12)$$

$$\tilde{\delta}^2 I = \int_a^b \tilde{\delta}^2 F dx \quad (3.13)$$

$$\tilde{\delta}^n I = \int_a^b \tilde{\delta}^n F dx \quad (3.14)$$

From Eq. 3.11, we have

$$\Delta I = \tilde{\delta} I + \tilde{\delta}^2 I + \dots + \tilde{\delta}^n I + \dots \quad (3.15)$$

The Eq. 3.15 is the definition given for the total increment of the functional $I(u)$ when the dependent function changes from u to $(u + \delta u)$, which is in conformity with some literature related to Eq. 1.3 in Section 1, see Long et al [15], Gelfand and Fomin [8], and Mesterton-Gibbons [16].

First, let's consider

$$\delta(\delta I) = \tilde{\delta}(\tilde{\delta} I) = \delta \int_a^b \delta F dx = \int_a^b \delta^2 F dx = \delta^2 I = 2! \int_a^b \tilde{\delta}^2 F dx \quad (3.16)$$

From Eq. 3.13, we conclude that

$$\delta(\delta I) = \tilde{\delta}(\tilde{\delta} I) = 2! \tilde{\delta}^2 I \quad (3.17)$$

As discussed in Section 3.2, this Eq. 3.17 is awkward.

Further,

$$\tilde{\delta}^n I = \int_a^b \tilde{\delta}^n F dx = \frac{1}{n!} \int_a^b \delta^n F dx = \frac{1}{n!} \delta^n I \quad \text{when } n \geq 1 \quad (3.18)$$

$$\tilde{\delta}^{n-1} I = \frac{1}{(n-1)!} \delta^{n-1} I \quad \text{when } n \geq 2 \quad (3.19)$$

$$\begin{aligned} \delta(\tilde{\delta}^{n-1} I) &= \tilde{\delta}(\tilde{\delta}^{n-1} I) = \frac{\delta(\delta^{n-1} I)}{(n-1)!} \\ &= \frac{\delta^n I}{(n-1)!} = \frac{n!}{(n-1)!} \tilde{\delta}^n I \quad \text{when } n \geq 2 \end{aligned} \quad (3.20)$$

$$\underbrace{\delta\delta \dots (\delta I)}_{n \text{ times}} = \underbrace{\tilde{\delta}\tilde{\delta} \dots (\tilde{\delta} I)}_{n \text{ times}} = \delta^n I = n! \tilde{\delta}^n I \quad (3.21)$$

4 Summary of the results and discussions from the variational processes

The origin of calculus of variations occurred about the same time period when Newton and Leibniz invented calculus of functions. There are many similarities between the differential operator and variational operator and it would be tempting to compare the forms between the total increment of a functional caused by variation of the dependent variable and the total increment of a function caused by the change of independent variable. Let's look at the Taylor's series for a function $f(x)$ first.

For a function $f(x)$ with the independent variable changes from x to $x+\Delta x$, the total increment is gives as

$$\Delta f = f(x+\Delta x) - f(x) = f'(x)\Delta x + \frac{1}{2!}f''(x)(\Delta x)^2 + \dots + \frac{1}{n!}f^{(n)}(x)(\Delta x)^n + \dots \tag{4.1}$$

For a functional $I(u)$ caused by a variation of the dependent function, in Section 2 in conformity with Eq. 1.2, Eq. 2.31 gives

$$\Delta I(u) = \delta I + \frac{1}{2}\delta^2 I + \dots + \frac{1}{n!}\delta^n I + \dots \tag{4.2}$$

For a functional $I(u)$ caused by a variation of the dependent function, in Section 3 in conformity with Eq. 1.3, Eq. 3.15 gives

$$\Delta I(u) = \tilde{\delta} I + \tilde{\delta}^2 I + \dots + \tilde{\delta}^n I + \dots \tag{4.3}$$

Obviously compared with Eq. 4.3, Eq. 4.2 is closer to Eq. 4.1 in terms of forms. Also, from the calculus of functions, we

know that $\frac{d^2 f(x)}{dx^2} = \frac{d(\frac{df}{dx})}{dx}$ or $d^2 f(x) = d\left(\frac{df}{dx}\right)$, where d is the differential operator of a function. There seems no reason why $\delta(\delta I)$ should not equal $\delta^2 I$ if we adopt the use of the variational operator. But according to the definition of higher variations in conformity with Eq. 1.3, $\tilde{\delta}(\delta I)$ does not equal $\delta^2 I$, as shown by Eq. 3.17. Such an inconsistency can be traced or induced by the definition in Eq. 3.15 or Eq. 4.3. Actually when the first variation of F or I is defined, it will be natural to check whether the second variation can be obtained from the first variation by having a further variation, and higher variations follow similar ways (" $\delta\delta$ " approach).

From Eq. 4.2 and Eq. 4.3, we have

$$\Delta I = \delta I + \frac{1}{2}\delta^2 I + \dots + \frac{1}{n!}\delta^n I + \dots = \tilde{\delta} I + \tilde{\delta}^2 I + \dots + \tilde{\delta}^n I + \dots \tag{4.4}$$

Similarly, we have

$$\Delta F = \delta F + \frac{1}{2}\delta^2 F + \dots + \frac{1}{n!}\delta^n F + \dots = \tilde{\delta} F + \tilde{\delta}^2 F + \dots + \tilde{\delta}^n F + \dots \tag{4.5}$$

The comparison for the variational processes in Section 2 (δ^k) and Section 3 ($\tilde{\delta}^k$) is summarized in the Table 1, where some of the equations are obvious and shown without proof. It should be noted that by making use of Eq. 2.3, the second variations of I can be expressed by Eq. 4.6 and Eq. 4.7 without the use of δ , which is the similar case for higher (than 2) variations.

$$\begin{aligned} \delta^2 I &= \int_a^b \left(\frac{\partial^2 F}{\partial u^2} \delta u^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u \delta u' + \frac{\partial^2 F}{\partial u'^2} \delta u'^2 \right) dx \\ &= \epsilon^2 \int_a^b \left(\frac{\partial^2 F}{\partial u^2} \eta^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \eta \eta' + \frac{\partial^2 F}{\partial u'^2} \delta \eta'^2 \right) dx \end{aligned} \tag{4.6}$$

$$\tilde{\delta}^2 I = \frac{1}{2!} \epsilon^2 \int_a^b \left(\frac{\partial^2 F}{\partial u^2} \eta^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \eta \eta' + \frac{\partial^2 F}{\partial u'^2} \delta \eta'^2 \right) dx \tag{4.7}$$

5 Conclusion

The fundamental variational process is reviewed in this paper, with a focus on the higher variation. Two definitions of higher variations of a functional (with the operator δ^n and $\tilde{\delta}^n$ respectively) are broadly found in the literature and they differ by a positive coefficient. The relationships between the two definitions for the higher variations are theoretically derived and summarized. When the two definitions of higher variations degenerate to the first variation, the difference disappears. It is found that the operator $\tilde{\delta}^n$ complies with the " $\delta\delta$ " approach, which means $\tilde{\delta}^2$ is equal to $\delta\delta$. However, the operator δ^n fails to comply with this. Therefore, the definitions for higher variations in terms of δ^n are strongly advised to be discarded. Although a specific functional form is used in this paper, the conclusion should not be restricted by this.

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Table 1. Summary of the results between δ^k and $\tilde{\delta}^k$

u, u', F, I $I = \int_a^b F(u, u', x)dx$	First variation ($k = 1$)	Second variation ($k = 2$)	n -th variation ($k = n \geq 2$)
u, u'	$\delta u = \tilde{\delta}u,$ $\delta u' = \tilde{\delta}u'$	$\delta^2 u = \tilde{\delta}^2 u =$ $\delta(\delta u) =$ $\tilde{\delta}(\tilde{\delta}u) = 0,$ $\delta^2 u' = \tilde{\delta}^2 u'$ $= \delta(\delta u')$ $= \tilde{\delta}(\tilde{\delta}u') = 0$	$\delta^n u = \tilde{\delta}^n u = 0,$ $\delta^n u' = \tilde{\delta}^n u' = 0$
F	$\delta F = \tilde{\delta}F$	$\delta^2 F = \delta(\delta F)$ $= \tilde{\delta}(\tilde{\delta}F)$ $= 2!\tilde{\delta}^2 F$	$\delta^n F = n!\tilde{\delta}^n F,$ $\delta(\tilde{\delta}^{n-1} F) = \tilde{\delta}(\tilde{\delta}^{n-1} F) =$ $\frac{n!}{(n-1)!}\tilde{\delta}^n F,$ $n \text{ times}$ $\underbrace{\delta\delta \dots (\delta F)}_{n \text{ times}}$ $= \tilde{\delta}\tilde{\delta} \dots (\tilde{\delta} F)$ $= \delta^n F = n!\tilde{\delta}^n F$
I	$\delta I = \tilde{\delta}I$	$\delta^2 I = \delta(\delta I)$ $= \tilde{\delta}(\tilde{\delta}I)$ $= 2!\tilde{\delta}^2 I$	$\delta^n I = n!\tilde{\delta}^n I,$ $\delta(\tilde{\delta}^{n-1} I) = \tilde{\delta}(\tilde{\delta}^{n-1} I) =$ $\frac{n!}{(n-1)!}\tilde{\delta}^n I,$ $n \text{ times}$ $\underbrace{\delta\delta \dots (\delta I)}_{n \text{ times}}$ $= \tilde{\delta}\tilde{\delta} \dots (\tilde{\delta} I)$ $= \delta^n I = n!\tilde{\delta}^n I$

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